

SOME LANDAU–GINZBURG MODELS VIEWED AS RATIONAL MAPS

E. BALLICO, E. GASPARIM, L. GRAMA, AND L. SAN MARTIN

ABSTRACT. [GGSM2] showed that height functions give adjoint orbits of semisimple Lie algebras the structure of symplectic Lefschetz fibrations (superpotential of the LG model in the language of mirror symmetry). We describe how to extend the superpotential to compactifications. Our results explore the geometry of the adjoint orbit from 2 points of view: algebraic geometry and Lie theory.

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1. MOTIVATION AND STATEMENT OF RESULTS

The data $W: X \rightarrow \mathbb{C}$ of a manifold together with a complex function is commonly known in the literature as a *Landau–Ginzburg model*; W is called the *superpotential*. Such LG models are fundamental ingredients to the study of questions motivated by the Homological Mirror Symmetry conjecture of Kontsevich [Ko]. When the superpotential is defined over a symplectic manifold, this involves the construction of a Fukaya–Seidel category. The objects of this category are associated to the singularities of W (Lagrangian vanishing cycles), see [Se]. These questions motivated us to construct examples of LG models and to study their symplectic geometry [GGSM1], [GGSM2]. Here we explore these examples from different points of view, namely, we are interested in the algebraic geometry and Lie theory of the smooth variety X .

Part of the homological mirror symmetry conjecture describes a duality between algebraic varieties and symplectic LG models. Subsequently Clarke [Cl] showed that the conjecture can be interpreted in further generality in such a way that both sides are LG models. According to whether

X is considered with a Kähler structure and algebraic potential or whether X is considered with a symplectic structure and holomorphic potential these are called *B-side LG model*, or *A-side LG model*, see [AAK] or [KKP]. In such terminology, the work of [GGSM1] provides a large family of examples of *A-side LG models*. Here we look at these examples from the point of view of *B-side LG models*.

Our constructions are carried out using Lie theory. Let \mathfrak{g} be a complex semisimple Lie algebra with Lie group G , and \mathfrak{h} the Cartan subalgebra. Consider the adjoint orbit $\mathcal{O}(H_0)$ of an element $H_0 \in \mathfrak{h}$, that is,

$$\mathcal{O}(H_0) := \{\text{Ad}(g)H_0, g \in G\}.$$

Let $H \in \mathfrak{h}$ be a regular element, and $\langle \cdot, \cdot \rangle$ the Cartan–Killing form. The main result of [GGSM1] shows that the height function

$$\begin{aligned} f_H: \mathcal{O}(H_0) &\rightarrow \mathbb{C} \\ X &\mapsto \langle X, H \rangle \end{aligned}$$

gives the orbit the structure of a symplectic Lefschetz fibration; thus corresponding to an *A-side LG model*. We show here that this height function can also be interpreted as a rational map on a projective compactification of $\mathcal{O}(H_0)$; hence corresponding to a *B-side LG model*.

In this work we restrict ourselves to the class of adjoint orbits which are diffeomorphic to cotangent bundles of projective spaces \mathbb{P}^n . This is the simplest case of semisimple orbit, yet already presenting somewhat surprising features. A harmonious combination of Lie theory and algebraic geometry happens naturally in this context, for example, we shall see that Lie theory provides rather efficient methods to identify the Segre embedding of a compactification of $\mathcal{O}(H_0)$. This method of carrying along Lie theory together with algebraic geometry is arguably where the core value of our contribution lies. We put forth the idea that there is much to profit from applying Lie theoretical methods to algebraic geometric problems. This work is a first instance of what we propose as a long term program. Certainly such combinations of the 2 areas have appeared in the literature in other contexts, the particularly new features of our contribution are the applications to the study Lefschetz fibrations and LG models.

Our results go as follows. We take $G = SL(n+1, \mathbb{C})$ and consider the adjoint orbit passing through $\mu = \text{Diag}(n, -1, \dots, -1)$, which we denote by \mathcal{O}_μ . The diffeomorphism type is then $\mathcal{O}_\mu \sim_{\text{dif}} T^*\mathbb{P}^n$. In Section 2 we recall the main result of [GGSM1] showing that the height function with respect to a regular element gives the adjoint orbit the structure of a symplectic Lefschetz fibration. We then describe the orbit and the regular fibers of f_H as affine varieties, and consider fibrewise compactifications. In section 3 for the case of $\mathfrak{sl}(2, \mathbb{C})$ we obtain:

Theorem. 3.1 *Let $X = \mathcal{O}_{(1, -1)}$, $H = \text{Diag}(1, -1)$ and $f_H: X \rightarrow \mathbb{C}$. Then f_H admits a fiberwise compactification with fibres isomorphic to \mathbb{P}^1 .*

In section 4 we present several ways to interpret the adjoint orbit, thus illustrating the interactions between Lie theory and algebraic geometry. In section 5 we describe (Zariski) open charts for \mathcal{O}_μ in terms of Bruhat cells.

Corollary. 5.4 *The domains of the parametrizations D_j corresponding to the Bruhat cells are open and dense in \mathcal{O}_μ .*

The orbit \mathcal{O}_μ is not compact, thus we must choose a compactification. Once again we are faced with the decision whether to use Lie theory or algebraic geometry. Recently in [GGSM2] a holomorphic open and dense embedding of \mathcal{O}_μ into $\mathbb{F} \times \mathbb{F}^*$ was constructed. Here, \mathbb{F} and \mathbb{F}^* represent a flag manifold and its dual flag, chosen such that $\mathcal{O}(H_0) \sim_{\text{dif}} T^*\mathbb{F}$ is a diffeomorphism. The immediate task that then follows is to extend the potential f_H to this compactification. Such an extension can not be made holomorphically, as explained in lemmas 6.1 and 6.2. We then proceed to extend the potential as a rational function. We consider here \mathcal{O}_μ as the adjoint orbit of $e_1 \otimes \varepsilon_1$ in $\mathbb{C}^n \times (\mathbb{C}^n)^*$. Set $V = \mathbb{C}^n$.

Theorem. 6.3 *The rational function on $V \otimes V^*$ that coincides with the potential f_H on $\mathcal{O}(v_0 \otimes \varepsilon_0)$ is given by*

$$R_H(A) = \frac{\text{tr}(A\rho_\mu(H))}{\text{tr}(A)}$$

for $A \in V \otimes V^* = \text{End}(V)$.

On the algebraic geometric side, [BCG] compactified the affine variety $X = \mathcal{O}_\mu$ to a projective variety \overline{X} by homogenising its defining ideal. Then the question begging to be asked is whether their algebraic compactification agrees with our Lie theoretical one. Using methods of computational algebraic geometry and a Macaulay2 algorithm [BG] identified a projective embedding of \overline{X} as the Segre embedding for the cases of $\mathfrak{sl}(n+1)$ with $n < 10$, and conjectured that the result holds true for all n . We provide an affirmative answer to this question, in particular concluding that the Lie theoretical compactification does coincide with the algebraic geometric one for the case of \mathcal{O}_μ .

Theorem. 7.1 *The embedding $\mathcal{O}_\mu \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{n*}$ obtained by Lie theoretical methods agrees with the Segre embedding obtained algebro-geometrically by homogenisation of the ideal cutting out the orbit \mathcal{O}_μ as an affine variety in $\mathfrak{sl}(n+1)$.*

Remark 1.1. Observe that the algebraic geometric method will in general produce singular compactifications, see [BCG, Sec. 6], whereas that the Lie theoretical method always embeds the orbit into a product of smooth flag manifolds.

We conclude the paper by presenting in section 7.1 the expressions of the Segre embedding and the rational potential R_H first for the case $n =$

3, with $\mu = (2, -1, -1)$ hence $\mathcal{O}_\mu \approx T^*\mathbb{P}^2$, and finally in general for $\mu = (n, -1, \dots, -1)$ when $\mathcal{O}_\mu \approx T^*\mathbb{P}^n$.

2. SYMPLECTIC LEFSCHETZ FIBRATIONS ON ADJOINT ORBITS

Let \mathfrak{g} be a complex semisimple Lie algebra with Cartan–Killing form $\langle X, Y \rangle = \text{tr}(\text{ad}(X)\text{ad}(Y)) \in \mathbb{C}$, and G a connected Lie group with Lie algebra \mathfrak{g} . Let $H_0 \in \mathfrak{h}$. The adjoint orbit of H_0 is defined as

$$\mathcal{O}(H_0) = \text{Ad}G \cdot H_0 = \{\text{Ad}(g)H_0 \in \mathfrak{g} : g \in G\}.$$

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a compact real form \mathfrak{u} of \mathfrak{g} . Associated to these subalgebras there are the subgroups $T = \langle \exp \mathfrak{h} \rangle = \exp \mathfrak{h}$ and $U = \langle \exp \mathfrak{u} \rangle = \exp \mathfrak{u}$. Denote by τ the conjugation associated to \mathfrak{u} , defined by $\tau(X) = X$ if $X \in \mathfrak{u}$ and $\tau(Y) = -Y$ if $Y \in i\mathfrak{u}$. Hence if $Z = X + iY \in \mathfrak{g}$ with $X, Y \in \mathfrak{u}$ then $\tau(X + iY) = X - iY$. In this case, $\mathcal{H}_\tau : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ defined by

$$(2.1) \quad \mathcal{H}_\tau(X, Y) = -\langle X, \tau Y \rangle$$

is a Hermitian form on \mathfrak{g} (see [SM, lemma 12.17]). We write the real and imaginary parts of \mathcal{H} as

$$\mathcal{H}(X, Y) = (X, Y) + i\Omega(X, Y) \quad X, Y \in \mathfrak{g}.$$

The real part (\cdot, \cdot) is an inner product and the imaginary part of Ω is a symplectic form on \mathfrak{g} . Indeed, we have

$$0 \neq i\mathcal{H}(X, X) = \mathcal{H}(iX, X) = i\Omega(iX, X),$$

that is, $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$, which shows that Ω is nondegenerate. Moreover, $d\Omega = 0$ because Ω is a constant bilinear form. The fact that $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$ guarantees that the restriction of Ω to any complex subspace of \mathfrak{g} is also nondegenerate.

Now, the tangent spaces to $\mathcal{O}(H_0)$ are complex vector subspaces of \mathfrak{g} . Therefore, the pullback of Ω by the inclusion $\mathcal{O}(H_0) \hookrightarrow \mathfrak{g}$ defines a symplectic form on $\mathcal{O}(H_0)$. With this choice of symplectic form, the main result of [GGSM1] says:

Theorem [GGSM1, Thm. 2.2] Let \mathfrak{h} be the Cartan subalgebra of a complex semisimple Lie algebra. Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_\mathbb{R}$ with H a regular element. The *height function* $f_H : \mathcal{O}(H_0) \rightarrow \mathbb{C}$ defined by

$$f_H(x) = \langle H, x \rangle \quad x \in \mathcal{O}(H_0)$$

has a finite number ($= |\mathcal{W}|/|\mathcal{W}_{H_0}|$) of isolated singularities and gives $\mathcal{O}(H_0)$ the structure of a symplectic Lefschetz fibration.

Remark 2.1. Note that the symplectic form Ω used here is not equivalent to the Kostant–Kirilov–Souriaux form on $\mathcal{O}(H_0)$.

3. ADJOINT ORBITS AS ALGEBRAIC VARIETIES

We now consider the case when $\mathfrak{g} = \mathfrak{sl}(n)$. To write down the adjoint orbit as an algebraic variety we can simply use the minimal polynomial corresponding to the diagonal matrix H_0 . Sometimes this method is not very economical, as it may give more equations than needed. In fact, using the entries of the minimal polynomial results in cutting out the adjoint orbit by n^2 equations inside the lie algebra $\mathfrak{sl}(n)$ which has dimension $n^2 - 1$. Thus, for instance we will certainly have far too many equations whenever the orbit is a complete intersection. Nevertheless, there is the advantage that this method works in all cases and is easily programable into a computer algebra algorithm.

Once we have described the orbit as an affine variety, we then wish to compactify it and to identify the closure of a regular fibre under this compactification.

As an example, we discuss the case of $\mathfrak{sl}(2, \mathbb{C})$. Take

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Weyl group $\mathcal{W} \simeq S_2$ acts via conjugation by permutation matrices. The two singularities of the potential are thus H_0 and $-H_0$. In this section we prove:

Theorem 3.1. *Let $X = \mathcal{O}_{(1,-1)}$, $H = \text{Diag}(1, -1)$ and $f_H : X \rightarrow \mathbb{C}$. Then f_H admits a fiberwise compactification with fibres isomorphic to \mathbb{P}^1 .*

We describe the orbit as an affine variety embedded in \mathbb{C}^3 . Writing a general element $A \in \mathcal{O}(H_0)$ as

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

the characteristic polynomial of A is

$$-(x - \lambda)(x + \lambda) - yz = \det(A - \lambda I) = \lambda^2 - 1.$$

This implies that the orbit $\mathcal{O}(H_0) \subset \mathfrak{sl}(2, \mathbb{C}) \simeq \mathbb{C}^3$ is an affine variety X cut out by the equation

$$(3.1) \quad x^2 + yz - 1 = 0.$$

We can compactify this variety by homogenising eq. 3.1 and embedding X into the corresponding projective variety. This produces the surface cut out by $x^2 + yz - t^2 = 0$ in \mathbb{P}^3 .

The height function on $X = \mathcal{O}(H_0)$

$$f_H(A) = \text{tr } HA = \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 2x,$$

has critical values ± 2 .

Thus, 0 is a regular value, and we express the regular fibre X_0 as the affine variety in $\{(y, z) \in \mathbb{C}^2\}$ cut out by the equation

$$yz - 1 = 0,$$

since it must satisfy eq. 3.1 and $x = 0$. As with the orbit, we homogenise this equation and embed the fibre into the corresponding projective variety $\overline{X_0}$ cut out by the equations $x = 0$ and $yz - t^2 = 0$ in \mathbb{P}^3 . Consider the natural embedding given by

$$\begin{aligned} i: \quad X_0 &\rightarrow \overline{X_0} \subset \mathbb{P}^3 \\ (y, z) &\mapsto [0, y, z, 1]. \end{aligned}$$

Claim: $\overline{X_0} \simeq \mathbb{P}^1$ and $\overline{X_0} \setminus i(X_0) = \{[0, 1, 0, 0], [0, 0, 1, 0]\}$.

Proof. In fact, $\overline{X_0}$ is cut out in \mathbb{P}^3 by a degree 2 polynomial, so the isomorphism with \mathbb{P}^1 follows immediately from the degree-genus formula $g = (d-1)(d-2)/2$.

For the second part, given a point $P = [0, y, z, t] \in \mathbb{P}^3$, we have that $P \in \overline{X_0}$ if and only if satisfies the equation $yz = t^2$. We have two possibilities:

- If $t \neq 0$, then $yz \neq 0$ and we may rewrite $P = [0, y, \frac{t^2}{y}, t] = [0, \frac{y}{t}, \frac{t}{y}, 1]$ and we have that $P = i(y', z')$ for $y' = \frac{y}{t}$ and $z' = \frac{t}{y}$.
- Otherwise $t = 0$, and there are two such points in $\overline{X_0}$, they are $[0, 1, 0, 0]$ and $[0, 0, 1, 0]$, neither of which belong to $i(X_0)$.

□

4. SEVERAL INCARNATIONS OF THE ORBIT

There are various ways to interpret the adjoint orbit $\mathcal{O}(H_0)$ depending on the point of view best suited to a given problem.

- By definition 2 the adjoint orbit is contained in the Lie algebra \mathfrak{g} , and consists of all the points $\text{Ad } g \cdot H_0$ with $g \in G$.
- $\text{Ad}(G) \cdot H_0$ can also be interpreted as a quotient of the group G , identifying $\mathcal{O}(H_0)$ with the homogeneous space G/Z_{H_0} where Z_{H_0} is the centralizer of H_0 .
- Take the simple roots of \mathfrak{g} that have H_0 in their kernel, denote by \mathfrak{p}_0 the parabolic subalgebra they generate, and by P_0 the corresponding subgroup. The compact subgroup K of G cuts out the subadjoint orbit $\text{Ad}(K) \cdot H_0$, which can be identified with the flag manifold $\mathbb{F}_{H_0} = G/P_{H_0}$. [GGSM2, Sec. 2.2] showed that $\mathcal{O}(H_0)$ is diffeomorphic to the cotangent bundle $T^*\mathbb{F}_{H_0}$.
- From a Riemannian point of view this is also diffeomorphic to the tangent bundle $T\mathbb{F}_{H_0}$.
- In [GGSM1] the orbit $\mathcal{O}(H_0)$ is given the symplectic structure taken from the imaginary part of the Hermitian form inherited from \mathfrak{g} . With this choice, the flag \mathbb{F}_{H_0} is the Lagrangian in $\mathcal{O}(H_0) \simeq T^*\mathbb{F}_{H_0}$ and corresponds to the zero section of the cotangent bundle.

- As seen in section 3 the adjoint orbit is an affine algebraic variety.
- [GGSM2, sec. 4.2] showed that $\mathcal{O}(H_0)$ can be identified with the open orbit of the diagonal action of G on the product $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0}^*$.

Example 4.1. Consider the case when $\mathfrak{g} = \mathfrak{sl}(n+1)$ and

$$H_0 = \text{Diag}(n, -1, \dots, -1).$$

Then the corresponding parabolic subgroup is

$$P_{H_0} = \{A \in \mathfrak{sl}(n+1) : a_{i1} = 0, \text{ for all } 1 < i \leq n+1\}.$$

Hence, P_{H_0} consists of matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & * & \cdots & * \end{pmatrix}$$

where $*$ denotes any complex number.

The centralizer of H_0 is the subgroup

$$Z_{H_0} = \{A \in \mathfrak{sl}(n+1) : a_{i1} = 0 = a_{1i} \text{ for all } 1 < i \leq n+1\}.$$

Hence, Z_{H_0} consists of matrices of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & * & \cdots & * \end{pmatrix}.$$

Here $F_{H_0} = \mathbb{P}^n$ and $\mathcal{O}(H_0) \sim_{\text{dif}} T^*\mathbb{P}^n$.

Different choices of $H_0 \in \mathfrak{sl}(n+1)$ will in general lead to different flag manifolds F_{H_0} . The flags range from the one of largest dimension, the full flag $F(1, 2, \dots, n)$ of subspaces $V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}$ with $\dim V_i = i$ down to the case of only subspaces of dimension 1, namely the projective space \mathbb{P}^n . The latter is usually called the minimal flag. The full flag occurs when all eigenvalues of H_0 are distinct, whereas the minimal flag occurs for $H_0 = \text{Diag}(n, -1, \dots, -1)$.

Notation 4.2. We denote by \mathcal{O}_μ the adjoint orbit $\mathcal{O}(H_0)$ associated to the matrix $H_0 = \text{Diag}(n, -1, \dots, -1) \in \mathfrak{sl}(n+1)$. Thus, this is the orbit of the *minimal flag* \mathbb{P}^n of $\mathfrak{sl}(n+1)$.

In what follows we will restrict our attention to orbits \mathcal{O}_μ . For this case we have yet another incarnation of $\mathcal{O}(H_0)$, namely as the adjoint orbit of $e_1 \otimes \varepsilon_1$ in $\mathbb{C}^n \times (\mathbb{C}^n)^*$. To verify this, first notice that $e_1 \otimes \varepsilon_1$ corresponds to the matrix with 1 in the $(1, 1)$ entry and zeroes elsewhere. Hence, as a linear operator, the image is 1-dimensional (generated by e_1) and the kernel is $n-1$ -dimensional. The action of G encompasses all matrices with the same spectrum. The identification with the orbit \mathcal{O}_μ is made using the moment map, as described in detail in [GGSM2, Sec. 4]. It identifies

the eigenspace associated to 0 in $g \cdot (e_1 \otimes \epsilon_0)$ with the eigenspace associated to the eigenvalue n of $\text{Ad}GH_0 \in \mathcal{O}_\mu$, and analogously identifies the eigenspaces of dimension $n - 1$.

5. TOPOLOGY ON \mathcal{O}_μ

We wish to cover \mathcal{O}_μ by open sets which are natural from the Lie theory point of view; these will turn out to be Zariski open as well. To do so, we use the model for the adjoint orbit of $e_1 \otimes \epsilon_1$ in $\mathbb{C}^n \times (\mathbb{C}^n)^* = \text{End}(\mathbb{C})$, which is the set of projections from \mathbb{C}^n to subspaces of dimension 1. This model is similar to the one where the orbit is viewed inside the product $\mathbb{P}^{n-1} \times \text{Gr}_{n-1}(n)$, given as the set of pairs $([v], V)$ such that $v \notin V$. Such a pair corresponds to a projection over $[v]$ with kernel V .

Given $H_0 \in \mathfrak{h}_\mathbb{R}$, let

$$\mathfrak{n}_{H_0}^+ = \sum_{\alpha(H_0) > 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_{H_0}^- = \sum_{\alpha(H_0) < 0} \mathfrak{g}_\alpha$$

be the sums of the eigenspaces of $\text{ad}(H_0)$ associated to positive and negative eigenvalues, respectively. The subspaces $\mathfrak{n}_{H_0}^\pm$ are nilpotent subalgebras. The corresponding subgroups $N_{H_0}^\pm = \exp \mathfrak{n}_{H_0}^\pm$ are closed in G and

$$N_{H_0}^- Z_{H_0} N_{H_0}^+ = N_{H_0}^- N_{H_0}^+ Z_{H_0}$$

is open and dense in G , where Z_{H_0} is the centralizer of H_0 in G and the product

$$(y, x, h) \in N_{H_0}^- \times N_{H_0}^+ \times Z_{H_0} \mapsto yxh \in N_{H_0}^- N_{H_0}^+ Z_{H_0}$$

is a diffeomorphism.

Consider now the adjoint orbit $\mathcal{O}(H_0) = \text{Ad}(G)H_0 = G/Z_{H_0}$. Then, the subset $\text{Ad}\left(N_{H_0}^- N_{H_0}^+ Z_{H_0}\right)H_0$, denoted simply by $N_{H_0}^- N_{H_0}^+ Z_{H_0} \cdot H_0$ satisfies

$$N_{H_0}^- N_{H_0}^+ Z_{H_0} \cdot H_0 = N_{H_0}^- N_{H_0}^+ \cdot H_0$$

since $Z_{H_0} \cdot H_0 = H_0$. Given that $N_{H_0}^- N_{H_0}^+ Z_{H_0}$ is open and dense in G it follows that $N_{H_0}^- N_{H_0}^+ \cdot H_0$ is open and dense in $\mathcal{O}(H_0)$. Moreover, on one hand the map

$$(y, x) \in N_{H_0}^- \times N_{H_0}^+ \mapsto yx \cdot H_0 \in N_{H_0}^- N_{H_0}^+ \cdot H_0$$

is a diffeomorphism, and on the other hand, $\exp : \mathfrak{n}_{H_0}^\pm \rightarrow N_{H_0}^\pm$ is a diffeomorphism. Therefore,

$$(Y, X) \in \mathfrak{n}_{H_0}^- \times \mathfrak{n}_{H_0}^+ \mapsto e^{\text{ad}(Y)} e^{\text{ad}(X)} \cdot H_0 \in N_{H_0}^- N_{H_0}^+ \cdot H_0$$

defines a coordinate system for $\mathcal{O}(H_0)$, around H_0 (whose codomain is open and dense).

The singularities of the Lefschetz fibration are $e_j \otimes \epsilon_j$, or equivalently, the points $w \cdot H_0$ with $w = (1j) \in \mathcal{W}$. For the singularity $e_j \otimes \epsilon_j = w \cdot H_0$ the algebras $\mathfrak{n}_{w \cdot H_0}^\pm$ and the groups $N_{w \cdot H_0}^\pm = \exp \mathfrak{n}_{w \cdot H_0}^\pm$ are given by

- $\mathfrak{n}_{w \cdot H_0}^+$ consists of the matrices with nonzero entries only at row j (0's in the diagonal) whereas $N_{w \cdot H_0}^+$ consists of the same matrices but with 1's in the diagonal.
- $\mathfrak{n}_{w \cdot H_0}^-$ consists of matrices with nonzero entries only at column j (0's in the diagonal), whereas $N_{w \cdot H_0}^-$ consists of the same matrices but with 1's in the diagonal.

For each index j the open Bruhat cell $\sigma_j = N_{(1j)H_0}^- [e_j]$ is the set of subspaces that are not contained in $V_j = \text{span}\{e_1, \dots, \widehat{e_j}, \dots, e_n\}$.

Proposition 5.1. *The domain of the coordinate system for index j is given by $N_{(1j)H_0}^- N_{(1j)H_0}^+ \cdot (1j)H_0 = \{([v], V) \in \mathbb{P}^{n-1} \times \text{Gr}_{\varepsilon-1}(n) : [v] \in \sigma_j, v \notin V\}$. This set coincides with the set of projections in the adjoint orbit of $e_1 \otimes \varepsilon_1$ whose image belongs to σ_j .*

Proof. $N_{(1j)H_0}^+ \cdot ([e_j], V_j) = \{[e_j]\} \cdot N_{(1j)H_0}^+$ where $N_{(1j)H_0}^+$ is the set of subspaces in $\text{Gr}_{n-1}(n)$ which do not contain e_j . Therefore if $n \in N_{(1j)H_0}^-$ then $n(\{[e_j]\} \cdot N_{(1j)H_0}^+ \cdot V_j)$ is the set of subspaces that do not contain $n[e_j]$. \square

Corollary 5.2. *The domain $D_j = N_{(1j)H_0}^- N_{(1j)H_0}^+ \cdot (1j)H_0$ of the chart for the index j is Zariski open.*

Proof. In the adjoint orbit $\mathcal{O} = \mathcal{O}(e_1 \otimes \varepsilon_1)$ of $e_1 \otimes \varepsilon_1$ in $\mathbb{C}^n \times (\mathbb{C}^n)^*$ the domain $N_{(1j)H_0}^- N_{(1j)H_0}^+ \cdot (1j)H_0$ is given by the elements $v \notin V_j$, that is, $\varepsilon_j(v) \neq 0$. Let $\mathcal{O}_k = \{v \otimes \varepsilon \in \mathcal{O} : \varepsilon(e_j) \neq 0\}$. Clearly $\mathcal{O} = \cup_k \mathcal{O}_k$ therefore $D_j = \cup_k D_j \cap \mathcal{O}_k$. However

$$\begin{aligned} D_j \cap \mathcal{O}_k &= \{v \otimes \varepsilon \in \mathcal{O} : \varepsilon_j(v) \otimes \varepsilon(e_k) = 0\} \\ &= \{v \otimes \varepsilon \in \mathcal{O} : \text{tr}((v \otimes \varepsilon)(e_k \otimes \varepsilon_j)) \neq 0\}. \end{aligned}$$

Since $v \otimes \varepsilon \in \mathcal{O} \mapsto \text{tr}((v \otimes \varepsilon)(e_k \otimes \varepsilon_j))$ is the restriction to \mathcal{O} of a linear map, it follows that $D_j \cap \mathcal{O}_k$ is Zariski open, and thus so is D_j . \square

Remark 5.3. Note that the complement of D_j is the set of zeros of the polynomial $\sum_k ((\varepsilon_j(v) \varepsilon(e_k))^2$.

We may restate corollary 5.2 as:

Corollary 5.4. *The domains of the parametrizations D_j corresponding the to Bruhat cells are open and dense in \mathcal{O}_μ .*

6. THE POTENTIAL VIEWED AS A RATIONAL MAP

Once the adjoint orbit has been compactified to a projective variety, we can no longer consider the potential as a holomorphic map, not even if we enlarge the target to \mathbb{P}^1 . For the case of the minimal flag, $\mathcal{O}(H_0)$ gets compactified to a product of projective spaces. The following 2 elementary lemmas show that the potential can not be extended holomorphically to the compactification.

Lemma 6.1. *Let $n > 1$. Then any holomorphic map $\mathbb{P}^n \rightarrow \mathbb{P}^1$ is constant.*

Proof. Consider a holomorphic map $g: \mathbb{P}^n \rightarrow \mathbb{P}^1$ and let $X_1 = g^{-1}(p_1)$ and $X_2 = g^{-1}(p_2)$ be two of its fibers. Then by Bezout's theorem $X_1 \cap X_2 \neq \emptyset$ therefore $p_1 = p_2$. \square

Lemma 6.2. *Let $n > 1$. Then any holomorphic map $\mathbb{P}^n \times \mathbb{P}^{n*} \rightarrow \mathbb{P}^1$ is constant.*

Proof. Suppose $f: \mathbb{P}^n \times \mathbb{P}^{n*} \rightarrow \mathbb{P}^1$ is holomorphic, and take $p \in \mathbb{P}^{n*}$, then the restriction $f|_{\mathbb{P}^n \times \{p\}}$ is holomorphic, thus constant by lemma 6.1. Hence, f factors through the projection $\mathbb{P}^n \times \mathbb{P}^{n*} \rightarrow \mathbb{P}^{n*}$ and induces a holomorphic map $\mathbb{P}^{n*} \rightarrow \mathbb{P}^1$ which also is constant by lemma 6.1. Thus f is constant. \square

Consequently, we aim to extend the potential to the projectivization as a rational map. This can be done as follows. Set $V = \mathbb{C}^n$.

Theorem 6.3. *The rational function on $V \otimes V^*$ that coincides with the potential f_H on $\mathcal{O}(v_0 \otimes \varepsilon_0)$ is given by*

$$R_H(A) = \frac{\text{tr}(A \rho_\mu(H))}{\text{tr}(A)}$$

for $A \in V \otimes V^* = \text{End}(V)$.

Proof. (1) Given 2 vector spaces V and W let $\mathbb{P}(V)$ and $\mathbb{P}(W)$ be the corresponding projective spaces. Then, $\mathbb{P}(V) \times \mathbb{P}(W)$ is in bijection with the subset of $\mathbb{P}(V \otimes W)$ of subspaces generated by decomposable elements $v \otimes w$, $v \in V$ and $w \in W$. The bijection is given by

$$(\langle v \rangle, \langle w \rangle) \in \mathbb{P}(V) \times \mathbb{P}(W) \mapsto \langle v \otimes w \rangle \in \mathbb{P}(V \otimes W).$$

(2) The flag \mathbb{F}_{H_μ} gets identified with the projective orbit of the space of maximal weight $V_\mu = \langle v_0 \rangle \in \mathbb{P}(V)$ (V = the representation space). The dual flag $\mathbb{F}_{H_\mu}^*$ gets identified to the projective orbit of the space of minimal weight $V_{\mu^*} = \langle \varepsilon_0 \rangle \in \mathbb{P}(V^*)$.

(3) The adjoint orbit $\mathcal{O}(H_\mu)$ gets identified with the open orbit in $\mathbb{F}_{H_\mu} \times \mathbb{F}_{H_\mu}^*$ by the diagonal action of G . Via the bijection of item 1 the compactification corresponds to the projectivization of the orbit $\mathcal{O}(v_0 \otimes \varepsilon_0)$ of $v_0 \otimes \varepsilon_0 \in V \otimes V^*$.

(4) The potential f_H on $\mathcal{O}(v_0 \otimes \varepsilon_0)$ can be written as

$$(6.1) \quad f_H(v \otimes \varepsilon) = \varepsilon(\rho_\mu(H) v) = \text{tr}((v \otimes \varepsilon) \rho_\mu(H))$$

where ρ_μ is the representation on V .

(5) The function f_H of (6.1) does not projectivize, that is, it does not induce a function on $\mathbb{P}(V \otimes V^*)$ since it is linear (homogeneous of degree 1). To projectivize the potential we must divide f_H by a linear function that is constant on the orbit $\mathcal{O}(v_0 \otimes \varepsilon_0)$, therefore obtaining a rational homogeneous function (of degree 0) which coincides with f_H on the orbit and projectivizes.

- (6) A linear function that can do the job is $\text{tr}(\nu \otimes \varepsilon) = \varepsilon(\nu)$. This linear functional is constant = 1 on $\mathcal{O}(\nu_0 \otimes \varepsilon_0)$, since if $\nu \otimes \varepsilon \in \mathcal{O}(\nu_0 \otimes \varepsilon_0)$ then there exists $g \in G$ such that

$$\nu \otimes \varepsilon = \rho_\mu(g) \nu_0 \otimes \rho_\mu^*(g) \varepsilon_0 = \rho_\mu(g) \nu_0 \otimes \varepsilon_0 \circ \rho_\mu(g^{-1}),$$

thus $\varepsilon(\nu) = \varepsilon_0 \circ \rho_\mu(g^{-1})(\rho_\mu(g) \nu_0) = \varepsilon_0(\nu_0) = 1$.

- (7) Therefore, the rational function on $V \otimes V^*$ that coincides with f_H on $\mathcal{O}(\nu_0 \otimes \varepsilon_0)$ and projectivizes is given by

$$R_H(A) = \frac{\text{tr}(A \rho_\mu(H))}{\text{tr}(A)}$$

for $A \in V \otimes V^* = \text{End}(V)$.

□

7. ALGEBRAIC COMPACTIFICATIONS AND THE CONJECTURE OF [BG]

The orbit \mathcal{O}_μ . We can also compactify \mathcal{O}_μ from an algebraic point of view. Let $X_n = \mathcal{O}(H_0)$ for $H_0 = \text{Diag}(n, -1, \dots, -1)$. Then A belongs to X_n if and only if it satisfies the equations of the minimal polynomial $(A - nI)(A + I) = 0$. To compactify to a projective variety \overline{X}_n we add an extra variable t and homogenise the equations to $(A - ntId)(A + tId) = 0$. The set $F_n = \overline{X}_n \setminus X_n$ has a complete description. Taking $t = 0$ we get that it is given by the system $A^2 = 0$. Set-theoretically in $\mathfrak{sl}(n+1, \mathbb{C})$ the system $A^2 = 0$ defines all trace-zero matrices such that t^2 divides the minimal polynomial. In [BG] it is proven that this algebraic compactification produces the Segre embedding (assuming the additional hypothesis of smoothness) and it is proven computationally using Macaulay 2 for the cases of $\mathfrak{sl}(n, \mathbb{C})$ for $n < 10$ that this algebraic compactification does produce the Segre embedding. However, it is left as a conjecture to show that this works in full generality.

The following result follows from the explicit calculations presented in the next section, and solves the conjecture of [BG] in the affirmative:

Theorem 7.1. *The embedding $\mathcal{O}_\mu \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{n*}$ obtained by Lie theoretical methods agrees with the Segre embedding obtained algebro-geometrically by homogenisation of the ideal cutting out the orbit \mathcal{O}_μ as an affine variety in $\mathfrak{sl}(n+1)$.*

Other adjoint orbits. As proved in [GGSM2, Sec. 3] the orbit of any trace zero diagonalizable matrix $H_0 \neq 0$ embeds as an open dense subset of $\mathbb{F} \times \mathbb{F}^*$ with \mathbb{F} a certain flag. This is the best compactification, but we wish to compare with other compactifications obtained via algebraic methods. To compactify algebraically [BCG] used the process of homogenization of the ideal defining the orbit inside its Lie algebra. A matrix A belongs to the adjoint orbit H_0 if and only if it satisfies the equations of the minimal polynomial of H_0 . Taking the entries of the minimal polynomial determines an ideal I cutting out $\mathcal{O}(H_0)$ as an affine variety inside

$\mathfrak{sl}(n, \mathbb{C})$. We may then obtain a compactification by homogenizing the ideal I . In general the resulting compactification will be very singular, see [BCG, Sec. 6]. So, it is not possible to generalize the conjecture of [BG] for all semisimple adjoint orbits.

8. LIE THEORETICAL COMPACTIFICATION AND THE SEGRE EMBEDDING

In this section we present the explicit Lie theoretical calculation of the Segre embedding, first with the case $n = 3$ and then the general case.

The case of $\mathfrak{sl}(3, \mathbb{C})$ Let $G = SL(3, \mathbb{C})$ and let g be an element of G . We write g as

$$(8.1) \quad g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$\det g = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} - a_{31}a_{13}a_{22} + a_{31}a_{23}a_{12} = 1.$$

The inverse of g is given by

$$(8.2) \quad g^{-1} = \begin{pmatrix} a_{33}a_{22} - a_{23}a_{32} & a_{13}a_{32} - a_{33}a_{12} & a_{23}a_{12} - a_{13}a_{22} \\ a_{31}a_{23} - a_{21}a_{33} & a_{11}a_{33} - a_{31}a_{13} & a_{21}a_{13} - a_{11}a_{23} \\ a_{21}a_{32} - a_{31}a_{22} & a_{31}a_{12} - a_{11}a_{32} & a_{11}a_{22} - a_{21}a_{12} \end{pmatrix}.$$

Let us describe the orbit $G \cdot (\nu_0 \otimes \varepsilon_0) = \rho(g)\nu_0 \otimes \rho^*(g)\varepsilon_0$. Recall that $\nu_0 = (1, 0, 0)^T$ and $\varepsilon_0 = (1, 0, 0)$. The actions are

$$(8.3) \quad \rho(g)\nu_0 = g\nu_0 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix},$$

and

$$(8.4) \quad \rho^*(g)\varepsilon_0 = \varepsilon_0 \circ g^{-1} = (a_{33}a_{22} - a_{23}a_{32} \quad a_{13}a_{32} - a_{33}a_{12} \quad a_{23}a_{12} - a_{13}a_{22}).$$

Therefore,

$$(8.5) \quad \rho(g)\nu_0 \otimes \rho^*(g)\varepsilon_0 = \begin{pmatrix} a_{11}a_{33}a_{22} - a_{11}a_{23}a_{32} & a_{11}a_{13}a_{32} - a_{11}a_{33}a_{12} & a_{11}a_{23}a_{12} - a_{11}a_{13}a_{22} \\ a_{21}a_{33}a_{22} - a_{21}a_{23}a_{32} & a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} & a_{21}a_{23}a_{12} - a_{21}a_{13}a_{22} \\ a_{31}a_{33}a_{22} - a_{31}a_{23}a_{32} & a_{31}a_{13}a_{32} - a_{31}a_{33}a_{12} & a_{31}a_{23}a_{12} - a_{31}a_{13}a_{22} \end{pmatrix}.$$

The eigenvalues of matrix (8.5) are:

- 1 associated to the vector $\mu = (a_{11}, a_{21}, a_{31})$ and
- 0 (zero) associated to the vectors $\xi = (a_{12}, a_{22}, a_{32})$ and $\eta = (a_{13}, a_{23}, a_{33})$.

Since the determinant of matrix (8.1) is nonzero, we have that the line generated by μ is transversal to the plane generated by ξ and η (this is the geometric description of the adjoint orbit). Using the moment map, we verify that the orbit of the tensor product is isomorphic to $\text{Ad}(G) \cdot H_0$, for H_0 chosen appropriately (as a multiple of $\text{Diag}(2, -1, -1)$).

Thus, we obtain an embedding of the minimal orbit

$$\varphi : G \cdot (\nu_0 \otimes \varepsilon_0) \rightarrow \mathbb{P}^2 \times G_2(\mathbb{C}^3),$$

given by $\varphi(g \cdot (\nu_0 \otimes \varepsilon_0)) = (\text{span}\{\mu\}, \text{span}\{\xi, \eta\})$.

An identification between $G_2(\mathbb{C}^3)$ and \mathbb{P}^2 is obtained by taking each 2-plane P in $G_2(\mathbb{C}^3)$ to the line ℓ_P generated by the normal vector P . Explicitly, if $P = \text{span}\{\xi, \eta\}$, then ℓ_P is generated by the vector

$$(8.6) \quad (a_{33}a_{22} - a_{23}a_{32})\vec{i} + (a_{13}a_{32} - a_{33}a_{21})\vec{j} + (a_{23}a_{12} - a_{13}a_{22})\vec{k} \\ = (a_{33}a_{22} - a_{23}a_{32}, a_{13}a_{32} - a_{33}a_{21}, a_{23}a_{12} - a_{13}a_{22}).$$

Observe that (8.6) recovers the result of (8.4).

Note that if a vector μ belongs to the plane generated by $\{\xi, \eta\}$ then μ is orthogonal to the vector described in (8.6), that is ,

$$(8.7) \quad (a_{11}, a_{21}, a_{31}) \cdot (a_{33}a_{22} - a_{23}a_{32}, a_{13}a_{32} - a_{33}a_{21}, a_{23}a_{12} - a_{13}a_{22}) = \\ -a_{11}a_{23}a_{32} + a_{11}a_{33}a_{22} + a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} - a_{31}a_{13}a_{22} + a_{31}a_{23}a_{12} = 0.$$

This expression corresponds to the determinant of a 3x3 matrix (contained in the complement of the orbit inside $\mathbb{P}^2 \times G_2(\mathbb{C}^3)$, that is, representing the case of a line contained in a plane).

Using the previous identification, we can now obtain the Segre embedding by taking the composite

$$(8.8) \quad G \cdot (\nu_0 \otimes \varepsilon_0) \rightarrow \mathbb{P}^2 \times G_2(\mathbb{C}^3) \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^8.$$

The image of $g \cdot (\nu_0 \otimes \varepsilon_0)$ by the composite (8.8) in \mathbb{P}^8 has homogeneous coordinates that are the same as the entries of matrix (8.5).

The rational map: We describe f_H , for $H = (3, -2, -1)$. (We chose this instead of $H = (1, 0, -1)$ to avoid the vanishing of monomials that would be caused by the zero, but the same method applied to any choice of H). The rational map R_H associated to f_H is given by

$$(8.9) \quad R_H(\nu \otimes \varepsilon) = \frac{\text{tr}((\nu \otimes \varepsilon)\rho(H))}{\text{tr}(\nu \otimes \varepsilon)} = \\ \frac{3a_{11}a_{33}a_{22} - 3a_{11}a_{23}a_{32} - 2a_{21}a_{13}a_{32} + 2a_{21}a_{33}a_{12} - a_{31}a_{23}a_{12} + a_{31}a_{13}a_{22} \\ a_{11}a_{33}a_{22} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} + a_{31}a_{23}a_{12} - a_{31}a_{13}a_{22}}{a_{11}a_{33}a_{22} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} + a_{31}a_{23}a_{12} - a_{31}a_{13}a_{22}}.$$

The denominator of (8.9) is the determinant, which equals 1 if the point belongs to the orbit, and vanishes if the point belongs to the complement of the orbit. Thus, for points in the orbit R_H coincides with f_H (up to a constant multiple). Consequently, we can use the composite (8.8) to define a map to \mathbb{P}^1 , factoring through the Segre embedding:

$$(8.10) \quad \mathbb{P}^2 \times G_2(\mathbb{C}^3) \rightarrow \mathbb{P}^1,$$

by

$$(8.11) \quad (\text{span}\{\mu\}, \text{span}\{\xi, \eta\}) \mapsto \\ [3a_{11}a_{33}a_{22} - 3a_{11}a_{23}a_{32} - 2a_{21}a_{13}a_{32} + 2a_{21}a_{33}a_{12} - a_{31}a_{23}a_{12} + a_{31}a_{13}a_{22} : \\ a_{11}a_{33}a_{22} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{33}a_{12} + a_{31}a_{23}a_{12} - a_{31}a_{13}a_{22}].$$

The general case: $\mathfrak{sl}(n, \mathbb{C})$

For an $n \times n$ matrix A , we denote by $A(i|j)$ the matrix obtained by removing the i -th row and the j -th column of A . Recall that the (i, j) -cofactor of A is the scalar

$$(8.12) \quad C_{ij} = (-1)^{i+j} \det A(i|j).$$

We denote by $C = C_{ij}$ the matrix of cofactors. The classical adjoint of A is the transpose of the matrix of cofactors:

$$(8.13) \quad (\text{adj } A)_{ij} = C_{ji}.$$

We will use the following 2 well known properties of the classical adjoint:

$$(8.14) \quad \sum_{i=1}^n A_{ik} (\text{adj } A)_{ji} = \delta_{kj} \det A;$$

$$(8.15) \quad A(\text{adj } A) = (\det A) \text{ id}.$$

In particular, for a fixed j we obtain $\sum_{i=1}^n A_{ij} (\text{adj } A)_{ji} = \det A$ (expansion in cofactors with respect to column j).

The general Segre embedding: Let $G = SL(n, \mathbb{C})$ and $g = (a_{ij}) \in G$. We denote by $w_i = (a_{1i}, \dots, a_{ni})$ the column vectors of g . Since $\det g = 1$, we have that $g^{-1} = \text{adj } g$.

Let $v_0 = (1, 0, \dots, 0) \in \mathbb{C}^n$ and $\varepsilon_0 = (1, 0, \dots, 0) \in (\mathbb{C}^n)^*$. We describe the orbit $G \cdot (v_0 \otimes \varepsilon_0)$. We have:

$$(8.16) \quad \rho(g) v_0 = g v_0 = (a_{11}, a_{21}, \dots, a_{n1}) = w_1;$$

$$(8.17) \quad \rho^*(g) \varepsilon_0 = \varepsilon_0 \circ g^{-1} = \varepsilon_0 \circ \text{adj } g = ((\text{adj } g)_{11}, (\text{adj } g)_{12}, \dots, (\text{adj } g)_{1n}).$$

Therefore,

$$(8.18) \quad \rho(g) v_0 \otimes \rho^*(g) \varepsilon_0 = M = M_{ij} = a_{i1} (\text{adj } g)_{1j}.$$

Observe that

$$(8.19) \quad \text{tr } M = \sum_{i=1}^n M_{ii} = \sum_{i=1}^n a_{i1} (\text{adj } g)_{1i} = \det g = 1.$$

We can describe explicitly the kernel and image of M :

$$\begin{aligned}
M(w_1) &= \begin{pmatrix} a_{11}(\text{adj } g)_{11} & a_{11}(\text{adj } g)_{12} & \dots & a_{11}(\text{adj } g)_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}(\text{adj } g)_{11} & a_{n1}(\text{adj } g)_{12} & \dots & a_{n1}(\text{adj } g)_{1n} \end{pmatrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}\{a_{11}(\text{adj } g)_{11} + a_{21}(\text{adj } g)_{12} + \dots + a_{n1}(\text{adj } g)_{1n}\} \\ \vdots \\ a_{n1}\{a_{11}(\text{adj } g)_{11} + a_{21}(\text{adj } g)_{12} + \dots + a_{n1}(\text{adj } g)_{1n}\} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = w_1.
\end{aligned}$$

Hence, w_1 is an eigenvector associated to the eigenvalue 1.

On the other hand,

$$\begin{aligned}
M(w_2) &= \begin{pmatrix} a_{11}(\text{adj } g)_{11} & a_{11}(\text{adj } g)_{12} & \dots & a_{11}(\text{adj } g)_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}(\text{adj } g)_{11} & a_{n1}(\text{adj } g)_{12} & \dots & a_{n1}(\text{adj } g)_{1n} \end{pmatrix} \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}\{a_{12}(\text{adj } g)_{11} + a_{22}(\text{adj } g)_{12} + \dots + a_{n2}(\text{adj } g)_{1n}\} \\ \vdots \\ a_{n1}\{a_{12}(\text{adj } g)_{11} + a_{22}(\text{adj } g)_{12} + \dots + a_{n2}(\text{adj } g)_{1n}\} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned}$$

Hence, w_2 is in the kernel of M . Analogously, we verify that w_2, \dots, w_n are in the kernel of M (and therefore they are eigenvectors associated to the zero eigenvalue). As a consequence, we obtain the embedding

$$(8.20) \quad \varphi : G \cdot (v_0 \otimes \varepsilon_0) \rightarrow \mathbb{P}(\mathbb{C}^n) \times G_{n-1}(\mathbb{C}^n)$$

given by $\varphi(g \cdot (v_0 \otimes \varepsilon_0)) = (\text{span}\{w_1\}, \text{span}\{w_2, \dots, w_n\})$.

Let P be the hyperplane generated by w_2, \dots, w_n . Denote by $\xi_P \in (\mathbb{C}^n)^*$ the linear functional associated to P (that is, P is in the kernel of ξ_P). Direct calculations show that

$$\xi_P = ((\text{adj } g)_{11}, (\text{adj } g)_{12}, \dots, (\text{adj } g)_{1n}).$$

The correspondence $P \mapsto \xi_P$ gives the identification

$$G_{n-1}(\mathbb{C}^n) \rightarrow \mathbb{P}((\mathbb{C}^n)^*).$$

The Segre embedding of the minimal orbit is the composite

$$(8.21) \quad G \cdot (v_0 \otimes \varepsilon_0) \rightarrow \mathbb{P}(\mathbb{C}^n) \times G_{n-1}(\mathbb{C}^n) \rightarrow \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}((\mathbb{C}^n)^*) \rightarrow \mathbb{P}^{n^2-1}.$$

The coordinates of the image of this composite in \mathbb{P}^{n^2-1} are the entries of matrix (8.18).

Observe that the complement of the adjoint orbit in $\mathbb{P}(\mathbb{C}^n) \times \mathbb{P}((\mathbb{C}^n)^*)$ is the *incidence correspondence variety* Σ'' (see [Ha, Ex. 6.12]) given by

$$(8.22) \quad \Sigma = \{(w, \xi) : \langle w, \xi \rangle = 0\} \subset \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}((\mathbb{C}^n)^*).$$

The rational map:

Let $H = \text{Diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}$, with $\lambda_1 > \dots > \lambda_n$ and $\lambda_1 + \dots + \lambda_n = 0$ (where \mathfrak{h} is the Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$).

We describe a rational map (factored through the Segre embedding), that coincides with the potential f_H on the adjoint orbit. Such a rational map is given by

$$(8.23) \quad \psi : \mathbb{P}^{n-1} \times G_{n-1}(\mathbb{C}^n) \rightarrow \mathbb{P}^1,$$

$$(8.24) \quad \psi([v], [\varepsilon]) = \frac{\text{tr}((v \otimes \varepsilon)\rho(H))}{\text{tr}(v \otimes \varepsilon)} = \frac{\sum_{i=1}^n \lambda_i a_{i1} (\text{adj } g)_{1i}}{\sum_{i=1}^n a_{i1} (\text{adj } g)_{1i}},$$

where the identification $([v], [\varepsilon]) \mapsto v \otimes \varepsilon$ is described in [GGSM2, Sec. 4.2]. Observe that if $([v], [\varepsilon])$ belongs to the adjoint orbit, then $\text{tr}(v \otimes \varepsilon) = 1$ (see eq. 8.19). Furthermore, the complement of the orbit is the *incidence correspondence variety* Σ' , that is, the set of pairs (ℓ, P) such that

$$(8.25) \quad 0 \subset \ell \subset P \subset \mathbb{C}^n,$$

where P is a hyperplane in \mathbb{C}^n and $\ell \subset P$ is a line. The variety Σ is classically denoted in Lie theory as the flag manifold $\mathbb{F}(1, n-1)$.

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